The Book of Fibrations: An Introduction To The Serre Spectral Sequence

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1 Introduction

The spectral sequence is a tool in homological algebra that finds wide application in algebraic topology and algebraic geometry.

They were discovered by Jean Leray in 1946 in the context of calculating a type of cohomology theory in algebraic geometry called sheaf cohomology, which is a cohomology theory on a structure called a sheaf. Leray attempted to make steps towards calculating the cohomology by instead examining the cohomology of a related sheaf called the pushforward sheaf. He found that the cohomology groups of the pushforward sheaf formed a cochain complex, so he took the cohomology. These cohomology groups again formed a cochain complex, so he took cohomology again. Eventually, after taking cohomology continuously, the groups converged to the cohomology of the sheaf.

Although this idea of constantly taking (co)homology may seem counterintuitive if we want to make actual calculations, we will see later that in practice this is not an issue if we construct our spectral sequence right.

We will talk about an important spectral sequence in algebraic topology called the Serre spectral sequence, which comes in both homological and cohomological varieties. Given a fibration $E \to B$ with fiber F, this spectral sequence relates the (co)homology of the total space E to the (co)homologies of B and F. This enables us to calculate the (co)homology of one of the spaces from the (co)homology of the two others. Given the wide variety of fibrations, the Serre spectral sequence can be a very powerful computational tool.

We will begin with a discussion of fibrations and their application to homotopy theory in Section 2. Then, we will proceed to give a qualitative feel for spectral sequences in Section 3, computing a couple of examples. In Section 4, we will do all of the homological algebra groundwork and construct the spectral sequence of a filtered chain complex. In Section 5, we will translate this to topology and construct the Serre spectral sequence. In Section 6, we will discuss the product structure of the cohomological Serre spectral sequence. We will end in Section 7 and perform several calculations of both homology and cohomology using our spectral sequence.

2 Fibrations

The *fiber bundle* is a useful construction in topology that can be used to provide compact descriptions of complicated spaces.

From the perspective of homotopy theory, however, it turns out that we don't need all of the nice properties of fiber bundles. In fact, the only property that we are interested in is the *homotopy lifting property*, which we define below.

Definition 2.1. A map $\pi : E \to B$ has the **homotopy lifting property with respect to** X if given a homotopy $f_t : X \times [0,1] \to B$ and a lift $\widetilde{f_0} : X \to E$ of $f_0 : X \to B$, there exists a homotopy lift $\widetilde{f_t} : X \times [0,1] \to E$ restricting to $\widetilde{f_0}$ on $X \times \{0\}$. This is illustrated in the diagram below, with the f_0 arrow implicit.

$$\begin{tikzcd} X \arrow[r, "\widetilde{f_0}"] \arrow[d, hook] & E \arrow[d, "\pi"] \\ X \times [0,1] \arrow[r, "f_t"] \arrow[ur, dotted, "\widetilde{f_t}"] & B \end{tikzcd}$$

We can verify as a quick example the homotopy lifting property for the fiber bundle $F \times B \to B$.

Example 2.2. The fiber bundle $F \times B \to B$ has the homotopy lifting property with respect to any space X.

Proof. Denote the components of $\widetilde{f_0}(x)$ in $F \times B$ by $(g_0(x), f_0(x))$.

Then we can define $\widetilde{f_t}(x) = (g_0(x), f_t(x))$ as a trivial lift. ■

We can also define homotopy lifting for pairs.

Definition 2.3. A map $\pi : E \to B$ has **homotopy lifting property with respect to** (X, A) if given a homotopy $f_t : X \times [0,1] \to B$ with a lift $\widetilde{g}_0 : X \to E$ of f_0 and a lift $\widetilde{g}_t : A \times [0,1] \to E$ of $f_t|_A$, there exists a homotopy lift $\widetilde{f_t} : X \times [0,1] \to E$ restricting to $\widetilde{g}_0$ and $\widetilde{g}_t$ on the appropriate domains.

We will continue by getting rid of fiber bundles and instead considering spaces called fibrations which have the homotopy lifting property with respect to some specified class of spaces X.

Definition 2.4. A **(Hurewicz) fibration** is a map $\pi : E \to B$ which satisfies the homotopy lifting property with respect to any topological space X. A **Serre fibration** satisfies the homotopy lifting property with respect to the disk D^n for any n.

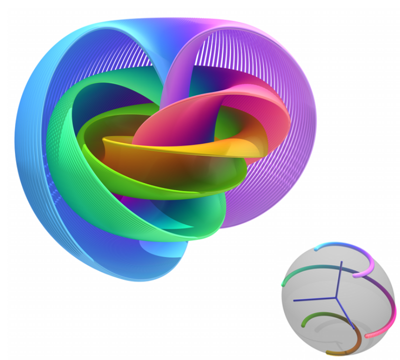
Due to the generality of their definition, fibrations come in all shapes and sizes. We showed above that the product space $F \times B \to B$ is a fibration, but we can come up with a couple of more nontrivial examples.

Example 2.5. We can construct a fibration $S^3 \to S^2$ with fiber S^1.

Identify S^3 with the unit ball $|z_0|^2 + |z_1|^2 = 1$ in $\mathbb{C}^2$. Noting that $\mathbb{CP}^1$ is homeomorphic to S^2, we can take the projection map $p : \mathbb{C}^2 - \{0\} \to \mathbb{CP}^1 \simeq S^2$ and restrict to S^3 to get a map $\pi : S^3 \to S^2$.

Picking a point in $\mathbb{S}^{\nvDash}$, its fiber in $\mathbb{C}^2$ is a line $\{(\lambda z_0, \lambda z_1) \mid \lambda \in \mathbb{C}\}$ for some fixed $z_0, z_1 \in \mathbb{C}$. Restricting that to S^3, the fiber is the solution set to the equation $|\lambda|^2(|z_0|^2 + |z_1|^2) = 1$, which is homeomorphic to S^1.

This fibration is known as the **Hopf fibration**, with a pretty artist's rendition depicted below. As we will see later in this section, it will allow us to easily calculate the value of $\pi_3(S^2)$.



Source: https://common.wikipedia.org/wiki/File:Hopf_Fibration.png

Example 2.6. There exists a fibration $\mathrm{SO}(3) \to S^2$ with fiber $\mathrm{SO}(2)$.

This fibration is produced by the natural action of $\mathrm{SO}(3)$ on S^2. The elements of $\mathrm{SO}(3)$ are parameterized by the pair (p, θ), where p is a point on S^2 and θ is the counterclockwise rotation angle of the element about the axis through p.

Our fibration is therefore just the projection map $(p, \theta) \to p$. The fiber of a point p is just the set of rotations that fix p. These are just rotations of the plane perpendicular to the axis through p, which is homeomorphic to $\mathrm{SO}(2)$.

Our next example is a much more general class of fibrations that we construct as a "twisted product". For any readers interested in category theory, this is just the pullback in the category of topological spaces.

Example 2.7. Given spaces X, Y, Z with maps $f : X \to Z$ and $g : Y \to Z$, the fiber product is a space $X \times_Z Y$ with projections $\pi_X : X \times_Z Y \to X$ and $\pi_Y : X \times_Z Y \to Y$ such that the following square commutes:

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z
\end{tikzcd}$$

Furthermore, for any other space W with projection maps $w_X : W \to X$ and $w_Y : W \to Y$ such that the square commutes, there is a unique map such that the following diagram commutes:

$$\begin{tikzcd}
W \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left, "w_Y"] \arrow[ddr, bend right, "w_X"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"] & Z
\end{tikzcd}$$

Concretely, we can define the fiber product as the set of pairs $(x, y) \in X \times Y$ such that $f(x) = g(y)$ with π_X and π_Y being the natural projections. Furthermore, it is not hard to see that these projections are fibrations.

This next example is a special example of the fiber product and is quite important. It shows that we can "transform" any continuous map into a fibration.

Example 2.8. Given a continuous map $f : X \to Y$, let X_f be the fiber product

$$\begin{tikzcd}
X_f \arrow[r] \arrow[d] & \mathrm{Hom}([0,1], Y) \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where the path space $\mathrm{Hom}([0, 1], Y)$ is given the compact-open topology and the corresponding map to Y is the end point map.

We can see immediately that X_f is homotopy equivalent to X by applying the concrete definition of fiber product. To see this, we can deformation retract X_f to the set $(x, \gamma_{f(x)}$, where $\gamma_{f(x)}$ is the constant path at $f(x)$. This is canonially homeomorphic to X, which gives our homotopy equivalence.

The space X_f is known as the **homotopy fiber** of f.

It is worth noting that in a fibration, the fibres can very significantly and are not always as nice as in our examples. However, if we have a path-connected base B, then it turns out that all the fibers are homotopy equivalent. This is a simple application of the homotopy lifting property and is a good exercise for the reader.

The below theorem serves as a strong example of the usefulness of the homotopy lifting property in homotopy theory.

Theorem 2.9. *Let $\pi : E \to B$ be a Serre fibration with B path-connected. Pick a basepoint $b_0 \in B$ and $x_0 \in \pi^{-1}(b_0) = F$. Then we have the following long exact sequence of homotopy groups*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to 0$$

where the map $\pi_n(E, x_0) \to \pi_n(B, b_0)$ is induced by π.

Before we prove this, we need to mention a useful property of Serre fibrations.

Lemma 2.10. *A Serre fibration has the homotopy lifting property for any CW pair (X, A).*

Proof. Lift cell by cell. ■

Now that we have a better understand of the lifting properties of a Serre fibration, we can prove the long exact sequence theorem.

Proof. This proof closely follows the one given in [Hatcher].

To show that this statement holds, we will show that $\pi_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$.

First, we show surjectivity. Pick a homotopy class in $\pi_n(B, b_0)$ represented by a map of pairs $f : (I^n, \partial I^n) \to (B, b_0)$.

Since f restricts to the constant map b_0 on ∂I^n, it does so as well on the subspace J^{n-1}. The constant map to x_0 out of J^{n-1} lifts $f|_{J^{n-1}}$. This gives us enough data to use the homotopy lifting property on the pair $(I^{n-1}, \partial I^{n-1})$ and get a lift $\widetilde{f} : I^n \to E$ of f.

Since $f(\partial I^n) = b_0$, we have $\widetilde{f}(\partial I^n) \subset \pi^{-1}(b_0) = F$. We also have $\widetilde{f}$ maps J^{n-1} to x_0, so its homotopy class lies in $\pi_n(E, F, x_0)$. By definition, $\pi(\widetilde{f}) = f$ and π is surjective.

To show injectivity, pick maps of triples $\widetilde{f}, \widetilde{g} : (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ such that $p\widetilde{f}$ and $p\widetilde{g}$ are homotopic.

Let $F : (I^n \times I, \partial I^n \times I) \to (B, b_0)$ be a homotopy from $p\widetilde{f}$ to $p\widetilde{g}$. Note that $\widetilde{f}$ and $\widetilde{g}$ give lifts of F on $I^n \times \{0\}$ and $I^n \times \{1\}$. The constant map to x_0 serves as a lift on $J^{n-1} \times I$. We again apply the homotopy lifting property with this data, this time to the pair $(I^n, \partial I^n)$ to get a lift $\widetilde{F} : (I^n \times I, \partial I^n \times I, J^{n-1} \times I) \to (E, F, x_0)$. This forms the desired homotopy between $\widetilde{f}$ and $\widetilde{g}$.

Most of the long exact sequence then follows from plugging in $\pi_n(B, b_0)$ for $\pi_n(E, F, x_0)$ in the homotopy long exact sequence of a pair.

To derive the end of the long exact sequence, we need to show that $\pi_0(F, x_0)$ surjects onto $\pi_0(E, x_0)$. This is where we use the path-connectedness of B. We need to show that F intersects any path-component of E. Given a point $x \in E$, we can construct a path from x to some point in F by taking a path from $\pi(x)$ to b_0 and then lifting. ■

As we can see, fibrations interact very nicely with homotopy groups and we can get a simple relation of the homotopy groups of the total space E with that of the base space B and the fiber F.

The rest of this paper will be devoted to understanding the relationships between the *homology* groups of E, B, and F. This relationship is considerably more complicated and requires the use of a computational tool called a *spectral sequence*.

3 What Is A Spectral Sequence?

In algebraic topology, we often make use of algebraic tools to compute homology groups. For example, we use short exact sequences like in the Universal Coefficient Theorem, or tools like the long exact sequence of a pair.

Spectral sequences are another such tool that, while complicated and containing a huge amount of data, often allow us to approximate difficult to calculate homology groups with other homology groups that are much easier to calculate.

We give a general definition below and then give the reader a feel for the concept with a couple of specific examples.

Definition 3.1. Let $\mathcal{A}$ be an abelian category. A **spectral sequence for homology** is a sequence of objects $E^i, E^{i+1}, \ldots$ with $i \geq 0$ and endomorphisms $d_r : E^r \to E^r$ satisfying the following two relations:

- $d_r \circ d_r = 0$
- $E^{r+1} = H(E^r) = \ker(d_r)/\operatorname{im}(d_r)$

The objects E^r are called the "pages" of a spectral sequence. This is because in practice they are generally bigraded modules or double complexes, which contain a lot of data and visually look like "pages" of modules.

Example 3.2. The simplest spectral sequence starts with the E^0 page equal to some chain complex C. Its differential d_0 is just the canonical boundary map on C.

Then we can start calculating. The E^1 page is the homology of C with respect to d, so it is a chain complex with terms $H_i(C)$. The only natural differential map here is the zero map.

As a result, we find that, by definition, the E^i page for $i \geq 2$ is equal to 0. In cases like these, we say that the spectral sequence **collapses** at E^1 and the output of the sequence is the homology groups of the chain complex.

Example 3.3. We can also use spectral sequences to calculate the homology of the total complex of a double complex.

A double complex C has modules $C_{p,q}$ with boundary maps $d_h : C_{p,q} \to C_{p-1,q}$ and $d_v : C_{p,q} \to C_{p,q-1}$ such that $d_h \circ d_h = d_v \circ d_v = 0$ and $d_h d_v = d_v d_h$.

We can then construct the total complex $T(C)$ of C as a chain complex satisfying $T(C)_n = \oplus_{p+q=n} C_{p,q}$. The boundary map $d_n : T(C)_n \to T(C)_{n-1}$ is defined on components $c_{p,q} \in C_{p,q}$ by $d_n(c_{p,q}) = d_v(c_{p,q}) + (-1)^q d_h(c_{p,q})$.

It turns out that the homology of the total complex arises from a spectral sequence. This is the most natural spectral sequence possible, which we start by setting the E^0 page to equal C.

We set the differential d_0 to be the vertical differential of C, depicted below.

$$
\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
C_{p-1,q+1} & C_{p,q+1} & C_{p+1,q+1} \\
\downarrow & \downarrow & \downarrow \\
C_{p-1,q} & C_{p,q} & C_{p+1,q} \\
\downarrow & \downarrow & \downarrow \\
C_{p-1,q-1} & C_{p,q-1} & C_{p+1,q-1} \\
\downarrow & \downarrow & \downarrow
\end{array}
$$

Therefore, the E^1 page has $E^1_{p,q} = H_q(C_{p,q})$. We then set the differential d_1 to be maps induced on homology by the horizontal differential. Since composition is preserved, we still have that $d_1 \circ d_1 = 0$, so we can take the homology going leftward.

$$\begin{tikzcd}
{} & E^1_{p-1,q+1} \arrow[l] & E^1_{p,q+1} \arrow[l] & E^1_{p+1,q+1} \arrow[l] & {} \arrow[l] \\
{} & E^1_{p-1,q} \arrow[l] & E^1_{p,q} \arrow[l] & E^1_{p+1,q} \arrow[l] & {} \arrow[l] \\
{} & E^1_{p-1,q-1} \arrow[l] & E^1_{p,q-1} \arrow[l] & E^1_{p+1,q-1} \arrow[l] & {} \arrow[l]
\end{tikzcd}$$

This gives us a page E^2 with $E^2_{p,q} = H_p(H_q(C_{p,q}))$. As a sneak peak, the diagram below shows how the differentials will be defined.

$$\begin{tikzcd}
E^2_{p-2,q+1} & E^2_{p-1,q+1} & E^2_{p,q+1} & E^2_{p+1,q+1} \\
E^2_{p-2,q} & E^2_{p-1,q} & E^2_{p,q} \arrow[ull] & E^2_{p+1,q} \arrow[ull] \\
E^2_{p-2,q-1} & E^2_{p-1,q-1} & E^2_{p,q-1} \arrow[ull] & E^2_{p+1,q-1} \arrow[ull]
\end{tikzcd}$$

In general, we will see that the differentials d_r will be defined as maps from $E^r_{p,q}$ to $E^r_{p-r,q+r-1}$. Note that these differentials, including the ones we described above, reduce the sum of the coordinates, or the "total degree" of the total complex, by 1. However, qualitatively, it seems like they are getting increasingly far away from their source $E^r_{p,q}$. In fact, since we have $E^r_{p,q} = 0$ for any $p < 0$ or $q < 0$, we find that the differentials at $E^r_{p,q}$ will vanish for sufficiently large r and $E^r_{p,q}$ will remain constant. We call this last module $E^\infty_{p,q}$ and bundle up all of them into the "E^∞" page. This geometric description of the differentials suggests why this spectral sequence is at all useful. Instead of trying to calculate the explicit differentials and resulting homology of the total complex, we calculate "approximate" differentials d_r that still decrease the degree of a chain by 1, but are easier to calcluate. Taking homology repeatedly acts as a way of refining our original group $E^0_{p,q}$. As we do this, we find that d_r gets "smaller" as r goes to ∞, and our limiting process then hopefully ends with $E^\infty_{p,q}$ equal to the homology of the total complex.

We will prove in the next section that there is a spectral sequence with first page $E^0 = C$ that satisfies $E^\infty_{p,q} = H_{p+q}(T(C))$.

Another thing that the reader may have noticed is that taking the vertical homology and then the horizontal homology was somewhat arbitrary. We will prove in the next section that taking the horizontal homology and then the vertical homology, along with the natural choice of direction for the generalized differentials, also converges to the homology of the total complex.

A fun exercise from [Vakil] makes use of this to prove the snake lemma.

Example 3.4. In the Snake Lemma, we are given the two exact rows (drawn to match our spectral sequence orientations)

$$\begin{tikzcd}
0 & C \arrow[l] \arrow[d, "\gamma"'] & B \arrow[l] \arrow[d, "\beta"'] & A \arrow[l] \arrow[d, "\alpha"'] & 0 \arrow[l] \\
0 & C' \arrow[l] & B' \arrow[l] & A' \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

and we want to show that there exists an exact sequence

$$0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma) \to 0$$

Plugging this into the double complex spectral sequence and taking the homology horizontally gives us that E^1 is 0 by exactness of the rows, and therefore E^∞ is 0.

Taking homology vertically, however, gives us the E^1 with the differentials drawn:

$$\begin{tikzcd}
0 & \ker(\gamma) \arrow[l] & \ker(\beta) \arrow[l] & \ker(\alpha) \arrow[l] & 0 \arrow[l] \\
0 & \operatorname{coker}(\gamma) \arrow[l] & \operatorname{coker}(\beta) \arrow[l] & \operatorname{coker}(\alpha) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

We can also draw the E^2 page:

$$\begin{tikzcd}
0 & H(\ker(\gamma)) & H(\ker(\beta)) & H(\ker(\alpha)) & 0 \\
0 & H(\operatorname{coker}(\gamma)) & H(\operatorname{coker}(\beta)) & H(\operatorname{coker}(\alpha)) \arrow[ull] & 0
\end{tikzcd}$$

We slightly abuse notation here to remind the reader how each E^2 module arose as the homology of an E^1 module. The arrow is the sole nonzero differential, so the spectral sequence stabilizes everywhere else. This immediately tells us that the homologies at $\ker(\alpha)$, $\ker(\beta)$, $\operatorname{coker}(\beta)$, and $\operatorname{coker}(\gamma)$ are 0. This gives us exactness of $0 \to \ker(\alpha) \to \ker(\beta) \to \ker(\gamma)$ and $\operatorname{coker}(\alpha) \to \operatorname{coker}(\beta) \to \operatorname{coker}(\gamma) \to 0$.

We also have that $H(\ker(\gamma))$ and $H(\operatorname{coker}(\alpha))$ must disappear since the differentials on the E^3 page are by definition identically 0. Therefore, the differential between them is an isomorphism. Unrolling the definition, this gives us that $\operatorname{coker}(\ker(\beta) \to \ker(\gamma)) = \ker(\operatorname{coker}(\alpha) \to \operatorname{coker}(\beta))$. This is exactly the condition that tells us that we can connect our two exact sequences to make the desired long exact sequence.

To drive the point home, we show the computational power of the double complex spectral sequence by proving the "twenty-five lemma", a 5×5 analogue of the nine lemma that would normally require a somewhat cryptic diagram chase to prove.

Example 3.5. Let's plug in the following diagram into the spectral sequence, where all the columns and all of the rows but the top one are exact.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_5 \arrow[r] \arrow[d] & A_4 \arrow[r] \arrow[d] & A_3 \arrow[r] \arrow[d] & A_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_5 \arrow[r] \arrow[d] & B_4 \arrow[r] \arrow[d] & B_3 \arrow[r] \arrow[d] & B_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_5 \arrow[r] \arrow[d] & C_4 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & D_5 \arrow[r] \arrow[d] & D_4 \arrow[r] \arrow[d] & D_3 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

By exactness of the columns, taking vertical homology tells us that the E^∞ page is 0.

Taking horizontal homology, all of the rows but the top row disappear, so the E^1 page just consists of the homology of the top row. Since the differentials from E^1 onwards are equal to 0, we find that $E^1 = E^\infty$ and therefore the homology of the top row must vanish as well and it is exact.

Note that this proof never actually uses the fact that our grid is 4×4. In fact, modulo typesetting time, we could have done this proof for an $N \times N$ grid for any large N!

4 Spectral Sequences of Filtered Complexes

Recall our example of the total complex above. We will define its differentials and prove it by constructing the spectral sequence of a more general construction called a **filtered complex**.

As motivation, consider results on the homology of a pair $A \subset X$ or triple $A \subset B \subset X$ of spaces. In both cases, we have a long exact sequence relating the homology of all of these spaces.

What if we tried to generalize this? Say we take a sequence $X_i \subset X_{i+1}$ of subspaces indexed by $\mathbb{Z}$. Their nth singular chain groups $C_n(X_i)$ by definition form an increasing sequence of submodules of $C_n(X)$. The singular chain complexes $C_*(X_i)$ piece together with the inclusion maps to form a grid of chain complexes. We will give a couple of definitions below to formalize this idea.

Definition 4.1. A **filtration** of a module M is a sequence of subspaces

$$\cdots \subset F_{-1}M \subset F_0M \subset F_1M \subset \ldots$$

indexed by $\mathbb{Z}$ such that $\cup_i F_iM = M$ and $\cap_i F_iM = \emptyset$.

Definition 4.2. A **filtered complex** is a chain complex C together with a filtration on each C_i such that the boundary maps preserve the filtrations, i.e. $\partial F_jC_i \subset F_jC_{i-1}$.

Often it serves us better to look at the graded structure of a filtration than the actual filtration itself. In nice cases, we can rebuild the original chain complex from its graded structure.

Definition 4.3. Given a module M with a filtration F, the **associated graded module** GM is equal to $\oplus_{i\in\mathbb{Z}} G_iM$, where $G_iM = F_iM/F_{i-1}M$.

This also has an analogue in filtered complexes.

Definition 4.4. Given a complex C with a filtration F, the **associated graded complex** G_iC is equal to the quotient of chain complexes $F_pC/F_{p-1}C$.

As a final note, observe that since the filtrations play nicely with the boundary map, a filtration on C induces a filtration on its homology. From this, we can talk about the graded pieces $G_pH_*(C)$ of its homology as well.

In the case of the homology of a pair, we are able to compute the homology $H(X)$ from the homologies $H(A)$ and $H(X/A)$, which correspond to the graded pieces of the filtration $A \subset X$.

Similarly, the idea here is to calculate the homology of a chain complex C by picking a filtration that has a nice graded structure with a homology that is easy to compute. A natural question to ask in the context of what we have gone over previously is if we can plug in the groups $H_*(G_pC)$ into a spectral sequence that gives us the groups $H_*(C)$ on the E^∞ page.

This turns out to be almost exactly correct.

Theorem 4.5. *Given a filtered complex C with filtration F, there is a spectral sequence E such that $E^\infty_{p,q} = G_pH_{p+q}(C)$.*

We say in this situation that E **abuts** to $H_{p+q}(C)$. A common shorthand that we will use later for this is $E^0_{p,q} \Rightarrow H_{p+q}(C)$.

We will start by constructing a few low-dimensional pages and differentials and then generalize.

The natural choice for the E^0 page is

$$E^0_{p,q} = G_pC_{p+q}$$

Each of the columns are the graded complexes. The choice of C_{p+q} instead of C_q is a strange one, but it will become clear why we chose it that way as we continue to derive the spectral sequence. The differential d_0 is just the regular boundary map $G_pC_{p+q} \to G_pC_{p+q-1}$. Therefore, the E^1 page satisfies

$$E^1_{p,q} = H_{p+q}(G_pC)$$

Given our goal at the E^∞ page, we can now see why the E^0 page was chosen the way it was. We will construct a series of different differentials d_r such that taking successive homology of these over $H_{p+q}(G_pC)$ will eventually "switch the grading" and give us $G_pH_{p+q}(C)$.

Therefore, $d_1 : E^1_{p,q} \to E^1_{p-1,q}$ is a map $H_{p+q}(G_pC) \to H_{p+q-1}(G_{p-1}C)$. The definition of d^1 follows from a bit of diagram chasing. Recall there is a short exact sequence of chain complexes $0 \to F_{p-1}C \to F_pC \to G_pC \to 0$. Therefore, taking the long exact sequence of homology gives us a natural map $H_{p+q}(G_pC) \to H_{p+q-1}(F_{p-1}C)$. Going along the long exact sequence for $0 \to F_{p-2}C \to F_{p-1}C \to G_{p-1}C \to 0$ gives us another map $H_{p+q-1}(F_{p-1}C) \to H_{p+q-1}(G_{p-1}C)$. We then take the composition to be d_1.

The modules $E^2_{p,q}$ are the homology with respect to d_1. Since they are a "homology of a homology", there is no nice formula for them like in the previous pages. However, we can lift any element $\alpha \in E^2_{p,q}$

to a representing element $a \in E^1_{p,q}$.

We make use of this to define the differential $d_2 : E^2_{p,q} \to E^2_{p-2,q+1}$. Given some $\alpha \in E^2_{p,q}$, lift it to $a \in E^1_{p,q} = H_{p+q}(G_pC)$. Using the LES of a pair, map it to $H_{p+q-1}(F_{p-1}C)$ again. Since α is a cycle with respect to d^1, by definition of d^1 we find that the image of a in $H_{p+q-1}(F_{p-1}C)$ is in the kernel of $H_{p+q-1}(F_{p-1}C) \to H_{p+q-1}(G_{p-1}C)$. By the LES of a pair, this is the image of $H_{p+q-1}(F_{p-2}C) \to H_{p+q-1}(F_{p-1}C)$, so there exists a lift $\widetilde{a}$ of a to $H_{p+q-1}(F_{p-2}C)$. We can then map this lift to $H_{p+q-1}(G_{p-2}C)$, project down to $E^2_{p-2,q+1}$, and denote the total composition of these maps by d_2.

This lifting property has a simple generalization to general r.

Lemma 4.6. *Given $\alpha \in E^r_{p,q}$, a representative $a \in H_{p+q}(G_pC)$, and its value $\delta a \in H_{p+q-1}(F_{p-1}C)$, there exists a lift $\widetilde{\delta a}$ of a in $H_{p+q-1}(F_{p-r}C)$.*

Proof. We do this by induction. The case for $r = 2$ has already been done above.

Assume that we have a lift $\widetilde{\delta a}'$ of a in $H_{p+q-1}(F_{p-r+1}C)$.

Since $\alpha \in E^r_{p,q}$, we find $\widetilde{\delta a}'$ is in the kernel of the map $H_{p+q-1}(F_{p-r+1}C) \to H_{p+q-1}(G_{p-r+1}C)$, which by the LES of a pair is equal to the image of the map $H_{p+q-1}(F_{p-r}C) \to H_{p+q-1}(F_{p-r+1}C)$. Therefore, we can pick $\widetilde{\delta a}$ to be some element in the preimage of $\widetilde{\delta a}'$. ■

We then define in general the differential d_r to be the composition:

$$d_r : E^r_{p,q} \to H_{p+q}(G_pC) \xrightarrow{\delta} H_{p+q-1}(F_{p-1}C) \to H_{p+q-1}(F_{p-r}C) \to H_{p+q-1}(G_{p-r}C) \to E^r_{p-r,q+r-1}$$

Note that we make a few arbitrary choices in this definition, so there are a couple of well-definedness conditions to check. Before we do that, we will attempt to get a feel for what these differentials mean as "approximations" of the homology group $G_pH_{p+q}(C)$.

First let us look at the differential d^1 at $E^1_{p,q}$.

Now we will check our well-definedness conditions. These are just long, inductive diagram chases. The reader should try to do them at least once, since they give a good understanding of the mysterious inner workings of the spectral sequence.

Lemma 4.7. *$d_r(\alpha)$ for $\alpha \in E^r_{p,q}$ is independent of choice of representative $a \in H_{p+q}(G_pC)$ and choice of lift $\widetilde{\delta a} \in H_{p+q-1}(F_{p-r}C)$*

Proof. We will prove this for d_2 and then induct.

Take a representative $a+b$ where a represents α and b represents a boundary $d_1\beta$. We have that by definition of d_1, b is in the image of $H_{p+q+1}(G_{p+1}C) \to H_{p+q}(F_pC) \to H_{p+q}(G_pC)$. Composing that with $H_{p+q}(G_pC) \xrightarrow{\delta} H_{p+q-1}(F_{p-1}C)$ is the zero map by exactness, so $\delta(a+b) = \delta a$.

Now we need to show that d_2 will be independent of the choice of lift of δa. Take two lifts $\widetilde{\delta a_1}$ and $\widetilde{\delta a_2}$. We have that $\widetilde{\delta a_1} - \widetilde{\delta a_2}$ lies in the kernel of the map $H_{p+q-1}(F_{p-2}C) \to H_{p+q-1}(F_{p-1}C)$. By exactness, this is in the image of $H_{p+q}(G_{p-1}C) \to H_{p+q-1}(F_{p-2}C)$. The image of $\widetilde{\delta a_1} - \widetilde{\delta a_2}$ under the map $H_{p+q-1}(F_{p-2}C) \to H_{p+q-1}(G_{p-2}C)$ is therefore by definition in the image of d^1 into $E^1_{p-2,q+1}$. After passing to $E^2_{p-2,q+1}$, this becomes zero and therefore d^2 is the same regardless of choice of lift.

Now we will prove this for d_r, assuming that well-definedness holds for all lower-order differentials.

By our inductive hypothesis, we can take our representative to be $a+b$ where a represents α and b represents a boundary $d_{r-1}\beta \in E^{r-1}$. Note that d_{r-1} also has the map $H_{p+q}(F_pC) \to H_{p+q}(G_pC)$ at the end, so composing with δ will take b to 0 as well.

Now we need to show that d_r is independent of the choice of lift of δa. Take two lifts $\widetilde{\delta a_1}$ and $\widetilde{\delta a_2}$. Mapping $\widetilde{\delta a_1}$ and $\widetilde{\delta_{a2}}$ into $H_{p+q-1}(F_{p-r+1}C)$, we find that they both map to lifts of δa to this group. By exactness, their image in $H_{p+q-1}(G_{p-r+1}C)$ is equal to 0.

∎

Next, we need to prove that the differentials compose with themselves to 0.

Lemma 4.8. *$d_r \circ d_r = 0$ for every r.*

Proof. Take an element $\alpha \in E^r_{p,q}$ and a representative a in $H_{p+q}(G_pC)$. Under the chain of maps that compose d_r, it gets sent to an element a' in $H_{p+q-1}(G_{p-r}C)$ which projects to $d_r\alpha$. Applying d_r again, this lifts to a' plus a boundary. In the proof of the previous lemma, we showed that this boundary disappears under the subsequent map $\delta : H_{p+q-1}(G_{p-r}C) \to H_{p+q-2}(F_{p-r-1}C)$. Also recall that a' is in the image of the map $H_{p+q-1}(F_{p-r}C) \to H_{p+q-1}(G_{p-r}C)$, so these two maps chain together by the LES of a pair and give us 0. ∎

Finally, we just need to prove that our spectral sequence converges to $G_pH_{p+q}(C)$. We are by default taking our filtration to satisfy $F_iC = 0$ for $i < 0$, so it is bounded like in the example of the double complex and $E^\infty_{p,q}$ exists.

Proof. Pick an element $\alpha \in E^\infty_{p,q}$ and its representative $a \in H_{p+q}(G_pC)$. Assuming a has image a_r in $E^r_{p,q}$, we require that $d_ra_r = 0$ for every r. By definition of d_r, this means that δa lifts to $H_{p+q-1}(F_{p-r}C)$ for every r. Since our filtration is 0 in negative indices and p is finite, we find that projecting the lift back down tells us that $\delta a = 0$.

Therefore, $a \in \ker(\delta)$, which by exactness implies that a is in the image of the map $H_{p+q}(F_pC) \to H_{p+q}(G_pC)$. Taking a lift $\overline{a}$, we project it via the map $H_{p+q}(F_pC) \to G_pH_{p+q}C = H_{p+q}(F_pC)/H_{p+q}(F_{p-1}C)$.

First, we show that this map is well-defined. Take two lifts $\overline{a}_1$ and $\overline{a}_2$. Their difference lies in the kernel of $H_{p+q}(F_pC) \to H_{p+q}(G_pC)$, which by exactness is in the image of $H_{p+q}(F_{p-1}C) \to H_{p+q}(F_pC)$. As a result, their difference goes to 0 under the quotient map.

Next, we need to show that this map is injective. Any element in the kernel must have a lift in the image of $H_{p+q}(F_{p-1}C) \to H_{p+q}(F_pC)$. This element is then recovered by mapping to $H_{p+q}(G_pC)$, which by exactness tells us that it is equal to 0.

Finally, surjectivity of this map follows by pulling back along the quotient map. Therefore, we have an isomorphism $E^\infty_{p,q} \simeq H_{p+q}(C)$. ■

The scenario of the filtered complex can be used to tackle many scenarios. To bring closure to this section, we will prove the convergence of the double complex spectral sequence, which is trivial after we have built up all of this machinery.

Corollary 4.9. *There exists a spectral sequence $E^0_{p,q} = C_{p,q} \Rightarrow H_{p+q}(T(C))$ for a double complex C.*

Proof. We can retrieve a filtered complex from $T(C)$ by filtering each $T(C)_n = \oplus_{p+q=n} C_{p,q}$ via its p-coordinate. Namely, we set $F_iT(C)_n = \oplus_{p+q=n, p\le i} C_{p,q}$.

By definition, we find the graded pieces satisfy $G_iT(C)_n = F_iT(C)_n/F_{i-1}T(C)_n = C_{i,n-i}$. Therefore, we find that our E^0 page is exactly the double complex C. It is easy to check that the first couple of differentials agree as well by unrolling the filtered complex definition.

Convergence to $H_{p+q}(T(C))$ is immediate by our filtered complex theorem.

Note that this proof is identical if we filter by q-coordinate. This proves the assertion we made way back in our "proofs" of the snake lemma and the sixteen lemma that the direction of the double complex spectral sequence does not matter. ■

5 The Serre Spectral Sequence

We can now construct a homological spectral sequence very important in homotopy theory known as the Serre (or Leray-Serre) spectral sequence.

Take $\pi : E \to B$ to be a Serre fibration with B a path-connected CW-complex with a fiber F.

Theorem 5.1. *If $\pi_1(B)$ acts trivially on the fibers, then there exists a spectral sequence satisfying $E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(E;\mathbb{Z})$.*

As we stated in the section on fibrations, the path-connected property of B ensures that the fibers are homotopy-equivalent and therefore the homology groups coincide, so this is well-defined.

We will prove this using the strategy from [Hutchings], constructing a filtered complex that produces the E^2 and E^∞ page that we want.

Proof. Let B^p be the p-skeleton of B. We can filter the integral singular chain complex $C_*(E)$ by defining $F_pC_*(E) = C_*(\pi^{-1}(B^p))$.

The graded pieces are $G_pC_*(E) = C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$. We can then calculate the E^1 page immediately:

$$E^1_{p,q} = H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$$

Next, the d^1 differential is defined as the composition

$$H_{p+q}(\pi^{-1}(B^p), \pi^{-1}(B^{p-1})) \to H_{p+q-1}(\pi^{-1}(B^{p-1})) \to H_{p+q-1}(\pi^{-1}(B^{p-1}), \pi^{-1}(B^{p-2}))$$

Now we are at the hardest part of the proof, namely showing that $E^1_{p,q}$ is equal to the cellular homology chain group $C^{CW}_p(B; H_q(F)) \simeq H_p(B^p, B^{p-1}) \otimes H_q(F)$.

Since $H_p(B^p, B^{p-1})$ is a free group over $\mathbb{Z}$ generated by the p-cells of B, this is isomorphic to $\oplus_\alpha H_q(F)$.

Let ϕ_α be the characteristic map for a p-cell D_α. We can construct a pullback square

$$\begin{tikzcd} \widetilde{D}_\alpha \arrow[r, "\phi^*_\alpha"] \arrow[d, "\xi_\alpha"] & E \arrow[d, "\pi"] \\ D_\alpha \arrow[r, "\phi_\alpha"] & B \end{tikzcd}$$

and set $\widetilde{S}_\alpha$ to be the preimage of the boundary S_α under ξ_α. We can put all of the ϕ^*_α together to form a map $\phi^* : \sqcup_\alpha(\widetilde{D}_\alpha, \widetilde{S}_\alpha) \to (\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$.

To show ϕ^* is an isomorphism on H_{p+q}, it suffices to show that excision holds since the RHS then just becomes the homology of a wedge sum of pullbacks of spheres. This follows by lifting the deformation retract $U \to B^{p-1}$ for some neighborhood U to a deformation retract $\pi^{-1}(U) \to \pi^{-1}(B^{p-1})$.

For the proof for $\oplus_\alpha H_{p+q}(\widetilde{D}_\alpha, \widetilde{S}_\alpha) \simeq \oplus_\alpha H_q(F)$, we defer the reader to [SSAT]. This proof is where the triviality of $\pi_1(B)$ assumption is used.

From these three isomorphisms, we get the desired isomorphism. Passing our map d^1 through these isomorphisms gives us the cellular boundary map. Finally, we can apply the fact that cellular and singular homology coincide on CW complexes and derive the E^2 page:

$$E^2_{p,q} = H_p(B, H_q(F))$$

■

6 The Cohomological Serre Spectral Sequence

Just like we have spectral sequences for homology, we can also define spectral sequences for cohomology. We often call this a **spectral sequence of cohomological type**. We state the cohomological versions of the two main results from above.

Theorem 6.1. *Given a filtered cochain complex C, there exists a spectral sequence $E_0^{p,q} = G_pC^{p+q} \Rightarrow H^{p+q}(C)$.*

Theorem 6.2. *Given a fibration $E \to B$ with B a path-connected CW-complex and a fiber F, there exists a spectral sequence $E_0^{p,q} = H^p(B; H^q(F)) \Rightarrow H^{p+q}(E)$.*

The cohomological Serre spectral sequence is often more powerful, since it has an internal product induced to the cup product on singular cohomology.

We will state the properties of this product without proof here, but refer the reader to [SSAT] for a more detailed discussion including proofs.

Theorem 6.3. *There exist bilinear products $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ satisfying the following properties:*

- *The product on E_2 is $(-1)^{qs}$ times the cup product $H^p(B; H^q(F)) \times H^s(B; H^t(F)) \to H^{p+q}(B; H^{q+t}(F))$ where multiplication of coefficients is induced by the cup product on $H^*(F)$.*

- *The differential d_r is a derivation satisfying $d_r(\alpha\beta) = d_r(\alpha)\beta + (-1)^{p+q}\alpha d_r(\beta)$. This induces a product on E_{r+1} from E_r.*

- *The cup product on $H^*(E)$ restricts to products on its filtered piecees, which in turn restrict to products on its graded pieces. This product coincides with the one on E^∞.*

7 Calculations

Now that we've waded through all of the messy homological algebra, it's time to do some interesting calculations with the Serre Spectral Sequence. We wield it in a manner similar to our long exact sequences in algebraic topology, namely to try and make as many of the terms as trivial as possible and then do our calculations from there.

Our first application is a pretty intuitively clear result on fibrations of spheres.

Proposition 7.1. *There is no fibration $S^m \to S^l \to S^n$ for $l \neq m + n$ or $m \neq n - 1$.*

Proof. We can just plug everything into the Serre spectral sequence.

By definition of the E^2 page, we have $E^2_{0,0} = E^2_{n,0} = E^2_{0,m} = E^2_{n,m} = \mathbb{Z}$ and $E^2_{p,q} = 0$ everywhere else.

Note that $E^2_{n,m}$ and $E^2_{0,0}$ will always have differentials equal to 0, so they are stable. By the homology of S^l, we require one of the nonzero groups to equal $H_l(S^l) = \mathbb{Z}$. Therefore, it is clear that we must have $l = m + n$.

Furthermore, we must have that the $E^2_{n,0}$ and $E^2_{0,m}$ groups must vanish. Therefore, we must have the differential $d^n : E^2_{n,0} \to E^2_{0,n-1}$ has codomain $E^2_{0,m}$. As a result, we have $m = n - 1$. ■

Our next application is in examining the loop space of a sphere. The suspension of a sphere ΣS^n is well-known, it is just S^{n+1}. However, the loop space ΩS^n is a little bit more mysterious. We can actually use the Serre spectral sequence to calculate its homology.

Proposition 7.2. *The integral homology $H_i(\Omega S^n)$ is equal to $\mathbb{Z}$ for i divisible by $n - 1$ and 0 otherwise.*

Proof. There exists a path-space fibration $PS^n \to S^n$ taking a path γ to its end-point, where PS^n is the space of all paths from some point $x \in S^n$. It is immediate that the fiber is the loop space ΩS^n.

Taking the Serre spectral sequence, the E^2 page has $E^2_{p,q} = H_q(\Omega S^n)$ for $p = 0, n$ and 0 otherwise.

The differentials remain at 0 until we hit the E^n page, where we have differentials $d^n : H_q(\Omega S^n) \to H_{q+n-1}(\Omega S^n)$.

$$
\begin{tikzcd}
\cdots & & \\
3n-3 & H_{3n-3}(\Omega S^n) & H_{3n-3}(\Omega S^n) \\
2n-2 & H_{2n-2}(\Omega S^n) & H_{2n-2}(\Omega S^n) \arrow[ul] \\
n-1 & H_{n-1}(\Omega S^n) & H_{n-1}(\Omega S^n) \arrow[ul] \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[ul] \\
 & 0 & n
\end{tikzcd}
$$

However, we observe that the path-space is contractible by retracting every path onto the constant map to the basepoint x. Therefore, the E^∞ page is 0 everywhere but $E^\infty_{p,q} = \mathbb{Z}$. Since the differentials are equal to 0 past E^n, we must have d^n is an isomorphism everywhere except for the ones going in and out of $E^2_{0,0}$. Therefore, since $H_0(\Omega S^n) = \mathbb{Z}$, its homology is equal to $\mathbb{Z}$ at every multiple of $n-1$. We have that for any $0 < k < n-1$ that $H_k(\Omega S^n) = E^2_{0,k} \simeq E^2_{n,k-n+1} = 0$, so homology is 0 everywhere else. ■

The next one, an exercise from [SSAT], involves working with the homotopy fiber.

Proposition 7.3. *Compute the homology of the homotopy fiber of a map $f : S^k \to S^k$ of degree n.*

Proof. The homotopy fiber is a fibration $F \to S^k_f \to S^k$.

We plug this into the Serre spectral sequence to get $E^2_{p,q}$ is equal to $H_q(F)$ for $p = 0, k$ and 0 otherwise. Since S^k_f is homotopy equivalent to S^k, we have the E^∞ page satisfies $E^\infty_{0,0} = E^\infty_{k,0} = \mathbb{Z}$.

The differentials are 0 until we hit E^k.

$$
\begin{tikzcd}
\cdots & & \\
3k-3 & H_{3k-3}(F) & H_{3k-3}(F) \\
2k-2 & H_{2k-2}(F) & H_{2k-2}(F) \arrow[ul] \\
k-1 & H_{k-1}(F) & H_{k-1}(F) \arrow[ul] \\
0 & \mathbb{Z} & \mathbb{Z} \arrow[ul] \\
 & 0 & k
\end{tikzcd}
$$

By our identity for E^∞, we require all of the differentials to be isomorphisms except the ones at $E^k_{0,0}$ and $E^k_{k,0}$, so the homology for F is periodic with period $k-1$ on the positive homology groups. This tells us that $H_i(F) = 0$ for $0 \le i < k-1$ and that $H_{j(k-1)} = H_{(j+1)(k-1)}$ for $j \ge 1$.

We have that by E^∞ the kernel of $d_k : E^k_{k,0} = \mathbb{Z} \to E^k_{0,k-1} = H_{k-1}(F)$ is isomorphic to $\mathbb{Z}$ and that this map is surjective. Therefore, we have $H_{k-1}(F) = \mathbb{Z}/n\mathbb{Z}$ for some n.

This is where we apply our degree idea. By the long exact sequence for homotopy on a fibration, we have that the sequence $0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \to \pi_{k-1}(F) \to 0$ is exact. Therefore, we find that $\pi_{k-1}(F) = \mathbb{Z}/n\mathbb{Z}$.

Applying the LES again tells us that all the lower homotopy groups of F vanish, so we find by the Hurewicz theorem that $H_{k-1}(F) = \mathbb{Z}/n\mathbb{Z}$ as well. ■

The product structure on the cohomological Serre spectral sequence also enables us to effortlessly calculate the cohomology rings of some Lie groups. These arguments follow the ones given in [Chicago].

Proposition 7.4. *$H^*(\mathrm{SU}(n)) = \bigwedge(x_3, x_5, \ldots, x_{2n-1})$, where we use $\bigwedge$ to denote the exterior algebra of the set of generators x_i with $|x_i| = i$.*

Proof. We will use the Serre spectral sequence on the fibration $\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1}$.

In the case that $n = 2$, this fibration degenerates to an isomorphism $\mathrm{SU}(2) \simeq S^3$. Therefore, $H^*(\mathrm{SU}(2)) = \bigwedge(x_3)$ by definition.

We then induct on n. Plugging into the Serre spectral sequence, the nonzero $E_2^{p,q}$ groups are at $p = 0, 2n-1$ and $q = 3, 5, \ldots, 2n-3$ by our inductive hypothesis. On these values, we have $E_2^{p,q} = H^q(\mathrm{SU}(n-1))$.

Let x be a generator of $E_2^{2n-1,0} = H^0(\mathrm{SU}(n-1)) = \mathbb{Z}$. First, we have $E_2^{0,i}$ are generated by x_i for $i \in \{3, 5, \ldots, 2n-3\}$.

To derive the generators of $E_2^{2n-1,i}$, we can take the product $E^{0,i} \times E^{2n-1,0} \times E^{2n-1,i}$. By definition, this is just the multiplication map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$, so $E_2^{2n-1,i}$ is generated by xx_i.

Furthermore, by the placement of the groups, we have all the differentials vanish and $E^2 = E^\infty$. It is easy to see that the cohomology of $\mathrm{SU}(n)$ vanishes above degree $2n-1$, so the cohomology ring becomes $\bigwedge(x_3, x_5, \ldots, x_{2n-3}, x)$ where $|x| = 2n-1$ as desired. ■

Proposition 7.5. *$H^*(\mathrm{U}(n)) = \bigwedge(x_1, x_3, \ldots, x_{2n-1})$.*

Proof. This is analogous to the proof above. We can take the fibration $U(n-1) \to U(n) \to S^{2n-1}$. Plugging in $n = 1$ gives us $U(1) \simeq S^1$, so the cohomology ring of $U(1)$ is $\bigwedge(x_1)$.

The resulting spectral sequence diagram, as can be seen from the fibration, is nearly identical and we have $E^2 = E^\infty$ in this case as well. ■

We conclude with a calculation of the cup product structure of $H^*(\Omega S^n)$ done similarly to its counterpart in [SSAT].

Proposition 7.6. *$H^*(\Omega S^n) = \begin{cases} \Gamma_{\mathbb{Z}}[x] & n \text{ is odd} \\ \bigwedge_{\mathbb{Z}}[x_{2k+1}] \otimes \Gamma_{\mathbb{Z}}[x_{2k}] & n \text{ is even} \end{cases}$*

Proof. We calculate the cohomology of ΩS^n with the path-space fibration to satisfy

$$H^i(\Omega S^n) = \begin{cases} \mathbb{Z} & n \text{ is divisible by } k-1 \\ 0 & \text{otherwise} \end{cases}$$

We can learn more by applying the product structure of the cohomology spectral sequence. Set x_i to be the generator of $E_2^{0,i(n-1)} = H^{i(n-1)}(\Omega S^n) = \mathbb{Z}$ for $i > 0$, and set x to be the generator of $E^{n,0} = H^0(\Omega S^n) = \mathbb{Z}$. By the product structure, we have that the other nonzero groups $E^{n,i(n-1)}$ are generated by xx_i.

Since the differentials d_n are isomorphisms, we have that $d_n(x_1) = x$ and $d_n(x_i) = x_{i-1}x$ for $i > 1$. Finally, the product x_ix is equal to the product xx_i since $(-1)^{i(n-1)n}$ is always equal to 1.

Now we will look at the derivation structure of the differential d_n to construct relations between our x_i. This is sign-dependent.

First assume n is odd. Then we have $d_n(x_1^2) = 2x_1d_n(x_1) = 2x_1x$. We also have $d_n(x_2) = x_1x$, so it follows that $x_1^2 = 2x_2$. In general, we can construct $d_n(x_1^i) = ix_1^{i-1}d_n(x_1) = ix_1^{i-1}x$. Assuming inductively that $x_1^{i-1} = (i-1)!x_{i-1}$, we have that $d_n(x_1^i) = i!x_{i-1}x = i!d_n(x_i)$, which tells us that $x_1^i = i!x_i$.

These relations tell us that the cohomology ring is the **divided polynomial algebra** $\Gamma_{\mathbb{Z}}$ over the set of x_i.

In the case that n is even, we now have the sign switched on our product. In this case, by the cup product we have $x_1^2 = -x_1^2 \to x_1^2 = 0$.

By induction, we can show $x_1x_{2k+1} = 0$ for every k. Taking the derivation d_n, we have it is equal to $xx_{2k+1} + x_1x_{2k}x = xx_{2k+1} - xx_1x_{2k}$. Now it suffices to show that $x_1x_{2k} = x_{2k+1}$. To see this, we take $d_n(x_1x_{2k}) = xx_{2k} - x_1x_{2k-1}x$. By induction, $x_1x_{2k-1} = 0$, so we are left with $xx_{2k} = d_n(x_{2k+1})$, so we get the required identity.

We can also show a divided polynomial algebra behavior in the even degrees. Assume that $x_2^k = k!x_{2k}$. The base case is trivial. Now we take $d_n(x_2^k) = kx_2^{k-1}d_n(x_2) = kx_2^{k-1}x_1x$. Applying our inductive hypothesis, we have this is equal to $k!x_{2k-2}x_1x = k!x_1x_{2k-2}x$. From our previous proof, $x_1x_{2k-2} = x_{2k-1}$, so we get $k!x_{2k-1}x = k!d_n(x_{2k})$, so $x_2^k = k!x_{2k}$ as desired.

We have an exterior algebra in the odd degree and a divided polynomial algebra in the even degree, which tells us our cohomology ring is the tensor product $\bigwedge_{\mathbb{Z}}[x_{2k+1}] \otimes \Gamma_{\mathbb{Z}}[x_{2k}]$. ■

8 Closing Remarks

Spectral sequences are an incredibly useful tool for computation thanks to their beautiful, almost geometric structure. However, as we have seen from some of the derivations in this paper, there is a lot of homological algebra and diagram chasing beneath the surface.

In addition, there are a couple of things we left out that the reader may want to look at. The first is the famous derivation of $\pi_4(S^3)$, which uses the Serre spectral sequence on the fibration $X \to S^3 \to K(\mathbb{Z}, 3)$. The second is the construction of spectral sequences via exact couples, which can be read about in [SSAT] or (with caution) on the nLab.

For anyone interested in homological algebra, the Grothendieck spectral sequence is an incredibly general spectral sequence that lets us compute the composition of derived functors.

The sources listed below are also great reads. Thanks for reading!

9 Bibliography

[Hatcher]: *Algebraic Topology* by Allen Hatcher (www.math.cornell.edu/∼hatcher/AT/AT.pdf)

[SSAT]: *Spectral Sequences in Algebraic Topology* by Allen Hatcher (www.math.cornell.edu/∼hatcher/SSAT)

[Hutchings]: *Introduction to Spectral Sequences* by Mike Hutchings (math.berkeley.edu/∼hutching/teach/215b-2011/ss.pdf)

[MIT]: MIT 18.906 Spring 2006 Lecture Notes (ocw.mit.edu/courses/mathematics/18-906-algebraic-topology-ii-spring-2006/lecture-notes)

[Chicago]: *The Cohomology of Lie Groups* by Jun Hou Fung (math.uchicago.edu/∼may/REU2012/REUPapers/Fung.pdf)

[Vakil]: *Spectral Sequences: Friend or Foe?* by Ravi Vakil (math.stanford.edu/∼vakil/0708-216/216ss.pdf)