An Introduction to Derived Functors

Rohil Prasad

December 11, 2015

Contents

1 Introduction

This paper is a short exposition to the basic theory of derived functors. The reader should be familiar with the basic language of category theory such as categories, functors, natural transformations, limits/colimits, monomorphisms/epimorphisms, and the category-theoretic definitions of kernel, cokernel, image, and quotient object. From the algebra side, they should know or be prepared to learn what an exact sequence of modules is.

All of the proofs in this paper were written up solely by the author, with inspiration derived from the sources listed in the bibliography.

In Section 2 we introduce abelian categories and exact functors, the basic objects of study in homological algebra. We continue our exposition of homological algebra in Section 3 and discuss chain complexes and homology. In Section 4 we introduce projective/injective resolutions and define the derived functor of a left/right exact functor.

2 Abelian Categories And Exact Functors

As one can guess from the name, an abelian category should be a category that "looks like" **Ab**. We will attempt to motivate the (somewhat complicated) definition of an abelian category by examining some nice properties that **Ab** has as a category.

The data of a category consists of its objects and its morphisms. We will begin our search by looking at the objects.

The only real '"object-focused" question we can ask about the objects of Ab is what can we do with them? Can we perform operations on abelian groups like adding or multiplying them? More generally, *what kinds of limits does the category of abelian groups have*? The answer to this question is pretty straightforward.

Proposition 2.1. **Ab** *has all small (co)limits.*

Proof. It suffices to show that **Ab** has all (co)products and (co)equalizers.

The product of two abelian groups G and H turns out to be the Cartesian product $G \times H$. The equalizer of two morphisms $f_1(g), f_2(g) : G \to H$ can be seen immediately as the subgroup $G' \hookrightarrow G$ such that $f_1(g) = f_2(g)$.

The constructions for coproduct and coequalizers are fairly simple as well. Constructing these and verifying that the examples above are in fact limits is a good exercise for a reader who has not seen this material before. ■

THe terminal and initial objects of **Ab** are particularly simple as well.

Proposition 2.2. *The zero group* $\mathbf{0}$ *is both terminal and initial in* **Ab**.

Proof. Immediate by definition. ■

Now, we will look at the morphisms in **Ab**. It turns out that the Hom-sets have a very nice structure.

Proposition 2.3. *Given two objects* $G, H \in \mathbf{Ab}$*, the set* $\mathrm{Hom}(G, H)$ *has the structure of an abelian group.*

Proof. Immediate by definition. ■

Now that we have gotten a bit more acclimated to **Ab**, we can write out the full definition of an abelian category below.

Definition 2.4. A category **A** is **abelian** if:

- Every Hom-set $\mathrm{Hom}(A, A')$ has the structure of an abelian group.
- It has all products and coproducts.
- It has all kernels and cokernels.
- Every monomorphism (resp. epimorphism) is the kernel (resp. cokernel) of another morphism.

As we can see, this definition contains within it some of the nice properties of **Ab** we discovered before. It is less clear why the bottom two properties are included, but one can interpret them as necessary to make the morphisms of **A** act more like the morphisms of **Ab**. In particular, the last property makes monomorphisms (resp. epimorphisms) act more like abelian group injections (resp. surjections).

Theorem 2.5. *(Freyd-Mitchell Embedding Theorem) Every abelian category* **A** *has a full, faithful embedding into the category* $\mathbf{R} - \mathbf{Mod}$ *of modules over some commutative ring* R.

Definition 2.6. A functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories is **additive** if the induced map $\mathrm{Hom}(A, A') \to \mathrm{Hom}(F(A), F(A'))$ is a homomorphism of abelian groups.

As we see from Freyd-Mitchell, we can carry over several concepts from $\mathbf{R}-\mathbf{Mod}$ to any abelian category. The theory of derived functors, in particular, is motivated by exact sequences and their interaction with additive functors.

Definition 2.7. An additive functor $F : \mathbf{A} \to \mathbf{B}$ is **exact** if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathbf{A}$, we find that the sequence $0 \to F(A) \to F(B) \to F(C) \to 0$ is exact as well.

If we only have $0 \to F(A) \to F(B) \to F(C)$ (resp. $F(A) \to F(B) \to F(C) \to 0$) is exact, then F is **left-exact** (resp. **right-exact**).

As we will see later, derived functors are actually a way of measuring how much a left/right-exact functor f fails to be exact.

3 Chain Complexes And Homology

Definition 3.1. A **chain complex** is a sequence

$$\dots \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \dots$$

of objects $\{C_i\}_{i\in\mathbb{N}}$ and morphisms $d_n : C_n \to C_{n-1}$ such that $d_i d_{i+1} = 0$ for every i. The morphisms d_i are called **boundary maps**.

We can also define maps between chain complexes.

Definition 3.2. A **chain map** $f : C \to D$ is a collection of morphisms $f_n : C_n \to D_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{i+1}"] & C_i \arrow[r, "d^C_i"] \arrow[d, "f_i"] & C_{i-1} \arrow[r, "d^C_{i-1}"] \arrow[d, "f_{i-1}"] & C_{i-2} \arrow[r, "d^C_{i-2}"] \arrow[d, "f_{i-2}"] & \cdots \\
\cdots \arrow[r, "d^D_{i+1}"] & D_i \arrow[r, "d^D_i"] & D_{i-1} \arrow[r, "d^D_{i-1}"] & D_{i-2} \arrow[r, "d^D_{i-2}"] & \cdots
\end{tikzcd}$$

It is easy to see from the above definition that we can compose chain maps and associativity holds. In fact, the chain complexes with chain maps over $\mathbf{A}$ form a category of chain complexes $\mathrm{Kom}(\mathbf{A})$.

Another important aspect of chain complexes is their homology.

Definition 3.3. The homology H_n of a chain complex C is the quotient object $\ker(d_n)/\operatorname{im}(d_{n+1})$.

This is well-defined by our condition that $d_n d_{n+1} = 0$. Note that by this definition, we can interpret an exact sequence as a chain complex with $H_n = 0$ for every n.

Another important fact is that homology is *functorial.*

Proposition 3.4. *Chain maps $f : C \to D$ preserve kernels and images of the boundary maps, and therefore induce maps $f_* : H_n(C) \to H_n(D)$ on homology.*

Proof. Left as an exercise for the reader. ■

Corollary 3.5. *The homology H_n is a functor* $\mathrm{Kom}(\mathbf{A}) \to \mathbf{A}$ *for every n.*

It is useful to be able to determine when two chain maps induce the same map on homology. One relation between the maps that we can look for is called **chain-homotopy**.

Definition 3.6. A **chain-homotopy** h from the chain map $f : C \to D$ to the chain map $g : C \to D$ is a collection of morphisms $h_n : C_n \to D_{n-1}$ that satisfy $h_{i-1}d_i^C + d_{i+1}^D h_i = f_i - g_i$ for every i:

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{i+1}"] & C_i \arrow[r, "d^C_i"] \arrow[d, shift right, "g_i"'] \arrow[d, shift left, "f_i"] & C_{i-1} \arrow[r, "d^C_{i-1}"] \arrow[dl, "h_{i-1}"'] \arrow[d, shift right, "g_{i-1}"'] \arrow[d, shift left, "f_{i-1}"] & C_{i-2} \arrow[r, "d^C_{i-2}"] \arrow[dl, "h_{i-2}"'] \arrow[d, shift right, "g_{i-2}"'] \arrow[d, shift left, "f_{i-2}"] & \cdots \\
\cdots \arrow[r, "d^D_{i+1}"] & D_i \arrow[r, "d^D_i"] & D_{i-1} \arrow[r, "d^D_{i-1}"] & D_{i-2} \arrow[r, "d^D_{i-2}"] & \cdots
\end{tikzcd}$$

Proposition 3.7. *Any two homotopic chain maps $f, g : C \to D$ induce the same map on homology.*

Proof. It suffices to show that the chain map $f - g$ is the zero map on homology.

Without loss of generality, let there be a homotopy h from f to g. The composition hd_C is equal to 0 on homology since the homology groups are quotients of the kernels of the boundary maps. The composition $d_D h$ lies inside $\operatorname{im}(d_D)$, which goes to 0 once we pass to homology groups.

Therefore, $f - g = hd_C + d_D h$ is the zero map on homology. ■

We can also define a dual concept to complexes called **cochain complexes**, in which the only formal difference is that the indices increase as in the example below:

$$\cdots \xrightarrow{d^{i-2}} C^{i-1} \xrightarrow{d^{i-1}} C^{i} \xrightarrow{d^{i}} C^{i+1} \xrightarrow{d^{i+1}} \cdots$$

Everything we introduced above has an analogue in cochain complexes. Namely, there is cohomology, cochain maps, cochain homotopy, and more.

4 Resolutions And Derived Functors

Exact sequences are powerful because they allow us to establish nice relations between objects. In particular, we can learn a lot about an object purely based on whether it fits into a certain exact sequence. As a simple example, consider an abelian group A such that the sequence $0 \to \mathbb{Z} \to A \to \mathbb{Z} \to 0$ is exact. It becomes immediate from this that A must equal the group $\mathbb{Z} \oplus \mathbb{Z}$.

This motivates the notion of a "resolution" of an object, where we put an object into an exact sequence of other objects that have some nice property. In our case, we will be looking at resolutions by injective/projective objects. We introduce the necessary definitions below.

Definition 4.1. An object A is **injective** if given any injective morphism $f : B_1 \to B_2$ and a morphism $g : B_1 \to A$, there exists a unique morphism $h : A \to B_2$ such that $g = hf$.

$$\begin{tikzcd} B_1 \arrow[r, "f"] \arrow[dr, "g"'] & B_2 \arrow[d, dashed, "\exists! h"] \\ & A \end{tikzcd}$$

A **projective object** is defined dually, switching the direction of the arrows and replacing "injective" with "surjective".

Definition 4.2. An **injective resolution** of an object A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \ldots$$

where all of the I_i are injective.

A **projective resolution** of A is an exact sequence

$$\ldots P_2' \to P_1' \to P_0' \to A \to 0$$

where all of the P_i' are projective.

We will often use the shorthand $0 \to A \to I$ to denote an injective resolution such as the one in the definition.

Definition 4.3. A category **C has enough injectives** if for any object C there exists an injective morphism from C to an injective object. Dually, our category **has enough projectives** if for any object C there exists a surjective morphism from a projective object to C.

Proposition 4.4. *Given an abelian category with enough injectives (resp. projectives), any object A has an injective (resp. projective) resolution.*

Proof. We prove the statement for injectives. Since our category has enough injectives, there is an injective morphism f from A to an injective object I^0. Then there is an injective morphism from the object I^0/A into an injective object I^1. We can then precompose this with the quotient map to create a map $f^0 : I^0 \to I^1$ with kernel equal to $\mathrm{im}(f) = A$.

We can then construct the rest of the I^i inductively. There is an injective map from $I^i/\,\mathrm{im}(f^{i-1})$ to I^{i+1}, which we precompose with the quotient map $I^i \to I^i/\,\mathrm{im}(f^i)$ to get a map $f^i : I^i \xrightarrow{f^{i+1}} I^{i+1}$. ■

Recall that exact sequences can be thought of as (co)chain complexes with (co)homology equal to 0. In particular, the increasing indices suggest to us that we can think of injective resolutions as cochain complexes.

From a category-theoretic perspective, a natural question to ask is if taking injective resolutions is "functorial" in some sense, i.e. whether a map between objects induces a natural map between their resolutions. We will see from the following two results that this is somewhat the case.

Proposition 4.5. *Given a morphism $g : A \to B$ of objects with injective resolutions $0 \to A \xrightarrow{f_A} I_A$ and $0 \to B \xrightarrow{f_B} I_B$, there exists a chain map between the resolutions that extends g, i.e. g is the chain map morphism from A to B.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f_A"] \arrow[d, "g"] & I_A^0 \arrow[r, "f_A^0"] \arrow[d, "g^0"] & I_A^1 \arrow[r, "f_A^1"] \arrow[d, "g^1"] & \dots \\
0 \arrow[r] & B \arrow[r, "f_B"] & I_B^0 \arrow[r, "f_B^0"] & I_B^1 \arrow[r, "f_B^1"] & \dots
\end{tikzcd}$$

Proof. By injectivity of the object I_B^0 and injectivity of the morphism $f : A \to I_A^0$, there exists a unique morphism $g^0 : I_A^0 \to I_B^0$ such that $g^0 f_A = f_B g$.

We can construct the other maps g^i inductively. We have an injective map $I_A^{i-1}/\ker(f_A^{i-1}) \to I_A^i$ induced by f_A^{i-1}.

We have that $g^{i-1}f_A^{i-2} = f_B^{i-2}g^{i-2}$, so they have the same image. From here, it is immediate that g^{i-2} induces a map from $\operatorname{im}(f_A^{i-2}) = \ker(f_A^{i-1})$ to $\operatorname{im}(f_B^{i-2}) = \ker(f_B^{i-1})$, which in turn induces a map h^{i-1} from $I_A^{i-1}/\ker(f_A^{i-1})$ to $I_B^{i-1}/\ker(f_B^{i-1})$ that commutes with the projections as shown:

$$\begin{tikzcd}
I_A^{i-1} \arrow[r, "g^{i-1}"] \arrow[d, two heads] & I_B^{i-1} \arrow[d, two heads] \\
\frac{I_A^{i-1}}{\ker(f_A^{i-1})} \arrow[r, "h^{i-1}"] & \frac{I_B^{i-1}}{\ker(f_B^{i-1})}
\end{tikzcd}$$

By injectivity of I_B^i, we get our desired g_B^i. ■

While we have proven that an extension exists, we have not proven that this extension is unique. This is not the case, but we do find a weaker "equivalence" of these chain maps in the concept of chain-homotopy that we introduced earlier.

Proposition 4.6. *Any two extensions of a map $g : A \to B$ to their respective injective resolutions $0 \to A \xrightarrow{f_A} I_A$ and $0 \to B \xrightarrow{f_B} I_B$ are chain-homotopic.*

Proof. Let our extensions be $g_1^i, g_2^i : I_A^i \to I_B^i$.

We will define $h^0 = 0$.

Next, we will construct $h^1 : I_A^1 \to I_B^0$. We have that $(g_1^0 - g_2^0)f_A = f_B(g - g) = 0$. Therefore, $g_1^0 - g_2^0$ factors through the cokernel $\operatorname{coker}(f_A)$. Since $\operatorname{im}(f_A) = \ker(f_A^0)$, we find that f_A^0 factors into a composition $I_A^0 \to \operatorname{coker}(f_A) \hookrightarrow I_A^1$, where the latter arrow is an injection.

Now we can apply the injectivity property of I_B^0 with the injection above and the map $\operatorname{coker}(f_A) \to I_B^0$ induced by $g_1^0 - g_2^0$ to get a unique map $h^1 : I_A^1 \to I_B^0$ that has the desired chain-homotopy property.

From here, we can construct h^i for general i inductively. We have the following derivation holds:

$$\begin{aligned}
(g_1^i - g_2^i - f_B^{i-1}h^i)f_A^{i-1} &= f_B^{i-1}g_1^{i-1} - f_B^{i-1}g_2^{i-1} - f_B^{i-1}h^i f_A^{i-1} \\
&= f_B^{i-1}(g_1^{i-1} - g_2^{i-1} - h^i f_A^{i-1}) \\
&= f_B^{i-1}f_B^{i-2}h^{i-1} \\
&= 0
\end{aligned}$$

Therefore, we have that the map $g_1^i - g_2^i - f_B^{i-1}h^i$ factors through the cokernel of f_A^{i-1}. The rest of the proof is identical to our derivation of h^1. ■

Now we finally have all the machinery necessary to define derived functors and prove some useful properties. Let $F : \mathbf{A} \to \mathbf{B}$ be a left-exact functor where $\mathbf{A}$ has enough injectives. Fix an object $A \in \mathbf{A}$ and pick an injective resolution $0 \to A \to I$. We can chop off A and then apply F to get the cochain complex $0 \to F(I^0) \to F(I^1) \to F(I^2) \to \dots$. Note that this is no longer exact, so it has nontrivial homology.

Definition 4.7. The **ith right-derived functor of F at A**, denoted by $R^iF(A)$, is the cohomology H^i of the cochain complex $0 \to F(I^0) \to F(I^1) \to F(I^2) \to \dots$.

Immediately we can see that there are some things "off" about this definition. First, we need to check well-definedness.

Proposition 4.8. *$R^iF(A)$ is the same regardless of choice of injective resolution of A.*

Proof. Pick two injective resolutions $A \to I_1$ and $A \to I_2$. ■

Second, we don't actually have a functor yet. The definition above is only on objects, but we find that there is a natural way to extend this to include morphisms as well.

Proposition 4.9. *R^iF is a functor $\mathbf{A} \to B$ sending $A \in \mathbf{A}$ to $R^iF(A)$ as defined above.*

Proof. This is immediate by construction of $R^iF(A)$.

We will denote the category of cochain complexes and cochain maps on an abelian category $\mathbf{C}$ by $\mathrm{coKom}(\mathbf{C})$.

Picking an injective resolution for each object in A along with the corresponding extensions of morphisms to these resolutions determines a functor $\mathbf{A} \to \mathrm{coKom}(\mathbf{A})$.

Chopping off the A term and applying F determines a functor $\mathrm{coKom}(\mathbf{A}) \to \mathrm{coKom}(\mathbf{B})$.

Then taking cohomology is the same as applying the functor H^i from $\mathrm{coKom}(\mathbf{B})$ to $\mathbf{B}$.

Our construction of R^iF is just the composition of these three functors, so it is a functor itself. By the previous lemma, two different choices of the first functor yield naturally isomorphic R^iF, so this choice does not matter. ■

Throughout this construction, the reader might have noticed that we never really used the fact that F is left-exact. As it turns out, the right-derived functors only have the nice properties described below if F is left-exact.

Proposition 4.10. *R^0F is naturally isomorphic to F.*

Proof. First, we show $R^0F(A) \simeq F(A)$ for every object A.

Taking an injective resolution $0 \to A \to I$, we find that $0 \to F(A) \to F(I^0) \to F(I^1)$ is exact by left-exactness of F. By definition, H^0 of $0 \to F(I^0) \to F(I^1) \to \dots$ is equal to the kernel of the map $F(I^0) \to F(I^1)$. By exactness of the previous sequence, this is equal to the image of $F(A) \to F(I^0)$, which is isomorphic to $F(A)$ as desired.

Naturality of these isomorphisms are left as an exercise to the reader. ■

Proposition 4.11. *If F is exact, then $R^iF = 0$ for $i > 0$.*

Proof. Take an object A and an injective resolution $0 \to A \xrightarrow{f} I$. We will show that the chain complex $F(I^0) \to F(I^1) \to \dots$ is exact.

Assume $i \geq 0$. Observe we have the short exact sequences $0 \to \ker(f^i) \to I^i \to \operatorname{im}(f^i) \to 0$, $0 \to \ker(f^{i+1}) = \operatorname{im}(f^i) \to I^{i+1} \to \operatorname{im}(f^{i+1}) \to 0$, and $0 \to \operatorname{im}(f^{i+1}) \to I^{i+2} \to \operatorname{coker}(f^{i+1}) \to 0$.

Since F is exact, it preserves all of these short exact sequences.

We find that $\operatorname{Im}(F(f^i))$ is equal to the composition $F(I^i) \to F(\operatorname{im}(f^i)) \to F(I^{i+1})$. The former map is surjective, so this is the same as the image of $F(\operatorname{im}(f^i)) \to F(I^{i+1})$. By our second exact sequence, this is equal to the kernel of $F(I^{i+1}) \to F(\operatorname{im}(f^{i+1}))$. By our third exact sequence, we can compose with the injection $F(\operatorname{im}(f^{i+1})) \to F(I^{i+2})$ and the kernel remains the same. However, the kernel of $F(I^{i+1}) \to F(\operatorname{im}(f^{i+1})) \to F(I^{i+2})$ is just equal to the kernel of $F(f^{i+1})$ by surjectivity of the former map and we have exactness at $F(I^{i+1})$ as desired.

By exactness of this sequence, the cohomology H^i for any $i > 0$ vanishes. ■

Recall the statement at the end of Section 1 that the right derived functors measured the failure of a left-exact functor to be exact. Precisely, it turns out that we can use derived functors to extend the exact sequence $0 \to F(A) \to F(B) \to F(C)$ into a long exact sequence.

Theorem 4.12. *Given an exact sequence $0 \to A \to B \to C \to 0$ of objects in $\mathbf{A}$, there is a long exact sequence*

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to \dots$$

The proof makes use of a few lemmas.

Lemma 4.13. *Exact functors preserve coproducts.*

Proof. Apply the exactness property to the sequence $0 \to A \to A \oplus B \to B \to 0$. ■

Lemma 4.14. *Any short exact sequence* $0 \to I' \to I \to I'' \to 0$ *of injectives splits.*

Proof. By injectivity of I' and the injection $I' \hookrightarrow I$, there is a unique morphism $I \to I'$ making the following diagram commute:

$$\begin{tikzcd} I' \arrow[r, hook] \arrow[dr, "\mathrm{id}"'] & I \arrow[d, dashed, "\exists!"] \\ & I' \end{tikzcd}$$

Note that we in fact only require I' to be injective. ■

Lemma 4.15. *(Snake Lemma)*

Any short exact sequence of cochain complexes $0 \to C' \to C \to C'' \to 0$ *induces a long exact sequence of cohomology*

$$0 \to H^0(C') \to H^0(C) \to H^0(C'') \to H^1(C') \to H^1(C) \to H^1(C'') \to \dots$$

Proof. Diagram chase. ■

Now we are ready to prove our big theorem.

Proof. Pick injective resolutions $A \to I_A$, $B \to I_B$, $C \to I_C$.

The maps in our exact sequence extend to chain maps between our injective resolutions, yielding an exact sequence $0 \to I_A \to I_B \to I_C \to 0$. Chopping off A, B, and C, we still have an exact sequence $0 \to I'_A \to I'_B \to I'_C \to 0$.

Taking F of this yields the exact sequence $0 \to F(I'_A) \to F(I'_B) \to F(I'_C) \to 0$. This exactness under F is immediate from the first two lemmas we proved above.

Applying the snake lemma then yields the desired long exact sequence of (co)homology. ■

5 Conclusion

We have built up the definition of derived functors from almost the ground up, and proven a couple of beautiful properties such as the long exact sequence. However, there is a lot more that we haven't covered.

In the interest of building up the theoretical foundations of derived functors as clearly and rigorously as possible, there is a noticeable absense of concrete examples. We will try to provide a jumping point here by recommending some constructions to look at.

If the reader is interested in algebraic topology, Hatcher's *Algebraic Topology* has a nice exposition on the Ext and Tor functors. These are the right-derived functor of the Hom functor and left-derived functor of the tensor product functor respectively.

If the reader is interested in algebraic geometry, then they may want to look at sheaf cohomology. This is the right-derived functor of the global section functor taking a sheaf to its global section.

If the reader is interested in algebraic number theory, then one application of derived functors is in the construction of group cohomology. This is the right-derived functor of the functor taking a G-module to the submodule of G-invariants.

Furthermore, the definition of derived functors given in this paper is not the most general one. For a more modern treatment of derived functors, which casts them in the light of another important homological algebra concept called the *derived category*, refer to the book "Methods of Homological Algebra" by Gelfand and Manin.

Bibliography

1. "Basic Category Theory" by Leinster

2. www.iiserpune.ac.in/ vmallick/2013s1/mth622/top_2_02_11.pdf

3. "Algebraic Topology" by Hatcher

4. "An Introduction to Homological Algebra" by Weibel

5. "Methods of Homological Algebra" by Gelfand and Manin